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# Synthetic Differential Supergeometry

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Three branches of modern mathematics make something of infinitesimals per se, namely, nonstandard analysis, synthetic differential geometry, and supergeometry. The first is concerned exclusively with invertible infinitesimals, whereas the second deals mainly with nilpotent ones. Both of the former two are engaged exclusively in commuting or bosonic infinitesimals, while the third treats anticommuting or fermionic ones, leading to so-called noncommutative mathematics. the unification of the first two approaches was nicely discussed by Moerdijk and Reyes, but the unification of the second and the third seems to remain open. The principal objective of this paper is to fill the gap, arguing that a super version of microlinear space, dubbed "supermicrolinear space," is a natural generalization of supermanifold, just as the synthetic concept of microlinear space is replacing the classical concept of smooth manifold. The central result of the paper is that the graded tangency of a supermicrolinear space forms a Lie superalgebra, while it is well known that the tangency of a microlinear space (i.e., its totality of vector fields) forms a Lie algebra.

## 0. INTRODUCTION

God created infinitesimals for mathematicians. Newton, Leibniz, and their contemporaries were so pious as to believe in them. However, most modern mathematicians are too atheistic and secular to accept them, preferring to try to couch every infinitesimal argument in their favorite $\varepsilon - \delta$ terms.

For all their efforts, modern mathematical iconoclasts have never succeeded in eradicating the shadows of infinitesimals. On the contrary, the tide has gradually been turning back for decades. The resurrection of infinitesimals can be seen in three arenas of modern mathematics, all of which have been developing for the latter half of this century. Leibniz' doctrine that differential and integral calculus should be developed within a number system containing infinitely small as well as infinitely large numbers besides finite ones was

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seriously taken up by Robinson in the 1960s. By using the techniques of model theory, he succeeded in constructing models of such number systems. His approach came to be known as nonstandard analysis, or more generally, as nonstandard mathematics, for which the reader is referred, e.g., to Stroyan and Luxemburg (1976).

Taking into account Grothendieck's treatment of nilpotent infinitesimals in his theory of schemes and Ehresmann's treatment of higher infinitesimals in his theory of jets, Lawvere proposed in the mid-1960s to build a formal differential geometry in which nilpotent and higher infinitesimals are coherently available in abundance. The enunciated geometry was known as synthetic differential geometry, in which such once-dubious expressions of standard differential geometry as "vector fields are infinitesimal transformations" retrieve their truly infinitesimal meanings. There microlinear spaces play the role of smooth manifolds in standard differential geometry. Calculation in coordinates in the standard theory of smooth manifolds is replaced by calculation of polynomials of infinitesimals followed by construction of corresponding quasi-colimit diagrams of small objects. Synthetic differential geometry is by no means a mere reformulation of standard differential geometry. What is a truism in the standard context often becomes a challenge in the synthetic context (e.g., the Jacobi identity of vector fields with respect to Lie brackets). For textbooks on synthetic differential geometry the reader is referred to Kock (1981), Lavendhomme (1996), and Moerdijk and Reyes (1991).

The last arena of modern mathematics in which we can witness the comeback of infinitesimals is supergeometry, championed by Berezin (1987), Kostant (1977), Manin (1988), and others, whose main concern was to provide a mathematical framework for bosons and fermions on an equal footing. Supergeometry lies at the entrance to noncommutative mathematics in the sense that the $\mathbb{Z}_2$-graded ring $\mathbb{R}$ of real supernumbers is not commutative, but only graded commutative. It borders on two principal branches of geometry, namely algebraic and differential geometries. The Russian school, inspired by the glory of algebraic geometry, prefers to get a super version of the smooth manifold by extending the structure sheaf of a smooth manifold to a sheaf of $\mathbb{Z}_2$-graded commutative algebras while retaining the smooth manifold itself as a supporting structure. On the other hand, DeWitt (1984), Rogers (1980, 1986), and others are biased in favor of differential geometry, replacing the set of real numbers by a Grassmann algebra and aping the theory of Banach manifolds. Physically speaking, the former approach retains the classical concept of space-time, but enlarges the set of observables, while the latter approach reconsiders the conventional notion of space-time itself. The reconciliation between the two approaches to supergeometry was discussed by Batchelor (1980). The two approaches were reviewed and elaborated in Bartocci *et al.* (1991).

The scope of nonstandard analysis has been confined to invertible infinitesimals. Although synthetic differential geometers have succeeded in dealing with invertible and nilpotent infinitesimals on an equal footing (Moerdijk and Reyes, 1991, Chapters VI and VII), they have been content with treating commuting or bosonic infinitesimals exclusively. However, if synthetic differential geometry is to cope with modern physics (e.g., modern physical theories of supergravity), it has to encompass anticommuting or fermionic infinitesimals besides commuting or bosonic ones. The principal concern of this paper is to take a first step in this direction, letting synthetic differential geometry deal with both commuting and anticommuting infinitesimals on an equal footing. Tangency is a more elusive matter in supergeometry than in bosonic geometry (Boyer and Gitler, 1984; Jadczyk and Pilch, 1981; Rothstein, 1986), and we believe that our synthetic approach to supergeometry could say something about it.

Just as microlinear spaces play a significant role in classical synthetic differential geometry, supermicrolinear spaces will occupy a fundamental position in our synthetic differential supergeometry. Calculation of polynomials of infinitesimals should be replaced by calculation of graded polynomials of infinitesimals. The general Kock axiom is to be replaced by its super version. These points will be discussed in Section 2. Section 1 reviews basic superalgebra, for which the reader is referred, e.g., to Leites (1980) or Manin (1988, Chapter 3). Section 3 is devoted to differential calculus, up to a simple super version of Taylor's formula. Tangency will be discussed in Section 4, though a proof of the main result that the totality of supervector fields on a supermicrolinear space forms a Lie superalgebra is relegated to the succeeding section. Section 5 is devoted to establishing the general graded anticommutativity and the general graded Jacobi identity as in Kock and Lavendhomme (1984) and Nishimura (1997b). In a subsequent paper (Nishimura, n.d.) we will discuss the semantic aspect of synthetic differential supergeometry in the spirit of Moerdijk and Reyes (1991).

As is usual in synthetic differential geometry, the reader should presume throughout the paper that we are working in a (not necessarily Boolean) topos, so that the excluded middle and Zorn's lemma have to be avoided. Objects of the topos go under such aliases as a "space," a "set," etc.

## 1. BASIC SUPERALGEBRA

Let $\mathbb{Z}$ denote the set of integers, whose elements are usually written $i, j, k, \ldots$, with or without subscripts. Let $\mathbb{Z}_2$ denote the set of integers mod 2, whose elements are usually written $\mathbf{p}, \mathbf{q}, \mathbf{r}, \ldots$, with or without subscripts.

We denote 0 mod 2 by $\mathbf{0}$, and 1 mod 2 by $\mathbf{1}$. Both $\mathbb{Z}$ and $\mathbb{Z}_2$ are commutative rings in standard sense.

A *superring* is a (not necessarily commutative) ring $\mathbb{R}$ with unity 1 whose underlying additive group is decomposed into two subgroups $\mathbb{R}_\mathbf{e}$ and $\mathbb{R}_\mathbf{o}$, called the *even* and *odd* parts of $\mathbb{R}$, respectively, such that:

(1.1) If $a$ and $b$ of $\mathbb{R}$ are both even or both odd, then $ab$ is even.
(1.2) Otherwise $ab$ is odd.

In short, a superring is a $\mathbb{Z}_2$-graded ring. Given a superring $\mathbb{R}$, we will often wrote $\mathbb{R}^\mathbf{0}$ for $\mathbb{R}_\mathbf{e}$ and $\mathbb{R}^\mathbf{1}$ for $\mathbb{R}_\mathbf{o}$. We say that $\mathbb{R}$ is *graded commutative* if for any $a \in \mathbb{R}^\mathbf{p}$ and any $b \in \mathbb{R}^\mathbf{q}$.

(1.3) $ab = (-1)^{\mathbf{pq}}ba$

Now we choose, once and for all, a graded commutative superring $\mathbb{R}$ intended to play the role of real numbers in our supermathematics. We have the following axiom:

(1.4) $\mathbb{R}$ is a graded commutative superring.

A *left* $\mathbb{R}$-*supermodule* is a left $\mathbb{R}$-module $M$ whose underlying abelian group is decomposed into even and odd parts $M_\mathbf{e}$ and $M_\mathbf{o}$ (also written $M^\mathbf{0}$ and $M^\mathbf{1}$) respectively, such that

(1.5) If $a \in \mathbb{R}^\mathbf{p}$ and $u \in M^\mathbf{q}$, then $au \in M^{\mathbf{p}+\mathbf{q}}$.

The notion of a *right* $\mathbb{R}$-*supermodule* is defined similarly. It is a truism that $\mathbb{R}$ can canonically be regarded as both left and right $\mathbb{R}$-supermodules. It is well known that every left $\mathbb{R}$-supermodule $M$ can be regarded as a right $\mathbb{R}$-supermodule in the sense that for any $a \in \mathbb{R}^\mathbf{p}$ and any $u \in M^\mathbf{q}$,

(1.6) $ua = (-1)^{\mathbf{pq}}au$

By the same token every right $\mathbb{R}$-supermodule can be regarded as a left $\mathbb{R}$-supermodule, so that we can feel free to use the term "$\mathbb{R}$-supermodule" without the "left" or "right." Each element $u$ of an $\mathbb{R}$-supermodule $M$ is decomposed uniquely into even and odd parts $u_\mathbf{e}$ and $u_\mathbf{o}$, so that $u = u_\mathbf{e} + u_\mathbf{o}$ with $u_\mathbf{e} \in M_\mathbf{e}$ and $u_\mathbf{o} \in M_\mathbf{o}$. If $u$ is even or odd, then it is called *pure*, with $|u|$ defined to be $\mathbf{0}$ or $\mathbf{1}$ according as $u \in M_\mathbf{e}$ or $u \in M_\mathbf{o}$.

An $\mathbb{R}$-*superalgebra* is an $\mathbb{R}$-algebra which is a superring and an $\mathbb{R}$-supermodule with respect to the same $\mathbb{Z}_2$-grading. An example of an $\mathbb{R}$-superalgebra is the totality of $\mathbb{R}$-valued functions on a set with componentwise operations, in which its even and odd elements are $\mathbb{R}_\mathbf{e}$-valued and $\mathbb{R}_\mathbf{o}$-valued ones. A *homomorphism of* $\mathbb{R}$-*superalgebras* is a homomorphism of their underlying $\mathbb{R}$-algebras preserving $\mathbb{Z}_2$-grading. Given two $\mathbb{R}$-superalgebras $A$

and $B$, we will often write $\mathrm{Spec}_B A$ for the set of homomorphisms of $\mathbb{R}$-superalgebras from $A$ to $B$.

The *polynomial* $\mathbb{R}$-*superalgebra* $\mathbb{R}[x_1, \ldots, x_n]$ *of variables* $x_1, \ldots, x_n$ with each of the variables being named as either even or odd is the graded commutative $\mathbb{R}$-superalgebra freely generated by $x_1, \ldots, x_n$ over $\mathbb{R}$. It is characterized by the following universal property (Manin, 1988, Chapter 3, §2, Item 5).

*Proposition 1.1.* For any graded commutative $\mathbb{R}$-superalgebra $A$ and any pure elements $a_1, \ldots, a_n$ of $A$ with $|a_i| = |x_i|$ $(1 \leq i \leq n)$, there exists a unique homomorphism $\varphi$ of $\mathbb{R}$-superalgebras from $\mathbb{R}[x_1, \ldots, x_n]$ to $A$ such that $\varphi(x_i) = a_i$ $(1 \leq i \leq n)$.

An ideal $I$ of an $\mathbb{R}$-superalgebra $A$ is called a *superideal of A* if both the even and odd parts of each element of $I$ belong to $I$.

A *Lie superalgebra* (*over* $\mathbb{R}$) is an $\mathbb{R}$-supermodule $L$ with a mapping $[\cdot, \cdot]\colon L \times L \to L$ such that for any $u, v, w \in L$ and any $a \in \mathbb{R}$:

(1.7) $[u + v, w] = [u, w] + [v, w]$
(1.8) $[au, v] = a[u, v]$
(1.9) $[u, v + w] = [u, v] + [u, w]$
(1.10) $[u, va] = [u, v]a$
(1.11) $[u, v] = (-1)^{|u||v|}[v, u]$ provided that both $u$ and $v$ are pure
(1.12) $[u, [v, w]] + (-1)^{|v|(|v|+|w|)}[v, [w, u]] + (-1)^{|w|(|u|+|v|)}[w, [u, v]] = 0$ provided that $u$, $v$, and $w$ are all pure

Conditions (1.11) and (1.12) are called the *graded anticommutativity* and the *graded Jacobi identity*, respectively.

Given an $\mathbb{R}$-superalgebra $A$, an *even* (*odd*, resp.) $\mathbb{R}$-*left-derivation* is an operation $X$ on $A$ acting from the left such that for any $u, v \in A$ and for any $a \in \mathbb{R}$,

(1.13) $X(u + v) = Xu + Xv$
(1.14) $X(ua) = (Xu)a$
(1.15) $X(uv) = (Xu)v + (-1)^{|u||X|}u(Xv)$ provided that $u$ is pure and $|X|$ is 0 or 1 according as $X$ is even or odd

An *even* (*odd*, resp.) $\mathbb{R}$-*right-derivation* is an operation $X$ on $A$ acting from the right such that for any $u, v \in A$ and for any $a \in \mathbb{R}$:

(1.16) $(u + v)X = uX + vX$
(1.17) $(au)X = a(uX)$
(1.18) $(uv)X = u(vX) + (-1)^{|v||X|}(uX)v$ provided that $u$ is pure and $|X|$ is 0 or 1 according as $X$ is even or odd.

## 2. WEIL SUPERALGEBRAS AND SUPERMICROLINEARITY

A *Weil superalgebra* is a local graded commutative $\mathbb{R}$-superalgebra $\mathcal{M}$ which, regarded as an $\mathbb{R}$-module, is finite-dimensional and can be written as $\mathcal{M} = \mathbb{R} + \mathfrak{m}$ (the first component is the $\mathbb{R}$-superalgebra structure and the second is the maximal superideal in $\mathcal{M}$). By way of example, the quotient superalgebra of the polynomial superalgebra $\mathbb{R}[x_1, \ldots, x_n]$ with respect to the superideal generated by $\{x_i x_j \mid 1 \leq i \leq n\}$ is a Weil superalgebra and is denoted by $\mathcal{M}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ with $\mathbf{p}_i = |x_i|$ $(1 \leq i \leq n)$. Given Weil superalgebras $\mathcal{M}_1$ and $\mathcal{M}_2$ with maximal superideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, a homomorphism of $\mathbb{R}$-superalgebras $\varphi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is said to be a *homomorphism of Weil superalgebras* if it preserves maximal superideals, i.e., if $\varphi(\mathfrak{m}_1) \subset \mathfrak{m}_2$. A finite limit diagram of $\mathbb{R}$-superalgebras is said to be a *good finite limit diagram of Weil superalgebras* if every object occurring in the diagram is a Weil superalgebra and every morphism occurring in the diagram is a homomorphism of Weil superalgebras. The diagram obtained from a good finite limit diagram of Weil superalgebras by taking $\mathrm{Spec}_{\mathbb{R}}$ is called a *quasi-colimit diagram of supersmall objects.*

The super version of the general Kock axiom, called the *general super-Kock axiom,* goes as follows:

(2.1) For any Weil superalgebra $\mathcal{M}$, the canonical $\mathbb{R}$-superalgebra homomorphism $\mathcal{M} \rightarrow \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathcal{M})}$ is an isomorphism.

Spaces of the form $\mathrm{Spec}_{\mathbb{R}}(\mathcal{M})$ for some Weil superalgebras $\mathcal{M}$ are called *superinfinitesimal spaces* or *supersmall objects*. The superinfinitesimal space corresponding to Weil superalgebra $\mathcal{M}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is denoted by $D(\mathrm{p}_1, \ldots, \mathrm{p}_n)$. In particular, $D$, $D(\mathbf{0})$, and $D(\mathbf{1})$ are denoted also by 1, $D$, and $D$, respectively. As examples, by Proposition 1.1, $D$, $D$, and $D(\mathbf{0}, \mathbf{1})$ are to be identified with $\{d \in \mathbb{R}_e \mid d^2 = 0\}$, $\{d \in \mathbb{R}_o \mid d^2 = 0\}$, and $\{(d_1, d_2) \in \mathbb{R}_e \times \mathbb{R}_o \mid d_1^2 = d_2^2 = d_1 d_2\}$, respectively. It is easy but interesting to see that $D = \mathbb{R}_o$, from which and the general super-Kock axiom it follows that every function from $\mathbb{R}_\mathbf{o}$ to $\mathbb{R}$ is linear (Dewitt, 1984, Exercise 1.1). Given $\mathbf{p} \in \mathbb{Z}_2$, $D^\mathbf{p}$ denotes $D$ or $D$ according as $\mathbf{p}$ is 0 or 1.

The superinfinitesimal space $D(\mathbf{0},\mathbf{1})$ will play a very important role in our discussion of tangency. First we note that $D(\mathbf{0}, \mathbf{1})$ can be identified with the subset of $\mathbb{R}$ consisting of all $d \in \mathbb{R}$ such that $d_\mathbf{e}^2 = d_\mathbf{o}^2 = d_\mathbf{e} d_\mathbf{o} = 0$. Under this identification $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$ corresponds to $d_1 + d_2 \in \mathbb{R}$. What concerns us most about $D(\mathbf{0}, \mathbf{1})$ is that the space $D(\mathbf{0}, \mathbf{1})$, regarded as a subset of $\mathbb{R}$, is closed under the left and right actions of $\mathbb{R}$ on itself, while $D$ and $D$ are not. More specifically, given $a \in \mathbb{R}$ and $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$, $a(d_1, d_2)$, and $(d_1, d_2)a$ go as follows:

(2.2) $a(d_1, d_2) = (a_\mathbf{e} d_1 + a_\mathbf{o} d_2, a_\mathbf{o} d_1 + a_\mathbf{e} d_2)$

(2.3) $(d_1, d_2)a = (d_1 a_{\mathbf{e}} + d_2 a_{\mathbf{o}}, d_1 a_{\mathbf{o}} + d_2 a_{\mathbf{e}})$

The following proposition is a simple, but intriguing application of the general super-Kock axiom.

*Proposition 2.1.* A function $f: D(\mathbf{0}, \mathbf{1}) \to \mathbb{R}$ is of the form $d \in D(\mathbf{0}, \mathbf{1}) \mapsto ad \in \mathbb{R}$ with some $a \in \mathbb{R}$ iff it satisfies the following condition:

(2.4) $f(db) = f(d)b$ for any $b \in \mathbb{R}$ and any $d \in D(\mathbf{0}, \mathbf{1})$

*Proof.* Trivially the only-if part obtains. If $f$ satisfies (2.4), then $f(0) = 0$, so that by the general super-Kock axiom there exist $a_1, a_2 \in \mathbb{R}$ such that

(2.5) $f(d_1, d_2) = a_1 d_1 + a_2 d_2$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$

We now consider two functions from $D(\mathbf{0}, \mathbf{1}) \times \overline{D}$ to $\mathbb{R}$, namely, $g_1$: $(d, e) \in D(\mathbf{0}, \mathbf{1}) \times D \mapsto f(de)$ and $g_2$: $(d, e) \in D(\mathbf{0}, \mathbf{1}) \times D \mapsto f(d)e$, which go as follows:

(2.6) $g_1(d_1, d_2, e) = a_1 d_2 e + a_2 d_1 e$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$ and any $e \in D$
(2.7) $g_2(d_1, d_2, e) = a_1 d_1 e + a_2 d_2 e$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$ and any $e \in D$

Since $g_1 = g_2$ by condition (2.4), $a_1 = a_2$ by the general super-Kock axiom, so that the desired $a$ is to be taken to be $a_1 = a_2$. This completes the proof. ■

The reader should note that $a$ in the above proposition is uniquely determined by the general super-Kock axiom. The proposition has the following important generalization.

*Proposition 2.2.* A function $f: D(\mathbf{0}, \mathbf{1})^n \to \mathbb{R}$ is of the form $(d_1, \ldots, d_n) \in D(\mathbf{0}, \mathbf{1})^n \mapsto a d_1 \ldots d_n$ with some $a \in \mathbb{R}$ iff it satisfies the following conditions:

(2.8) $f(d_1, \ldots, d_k b, d_{k+1}, \ldots, d_n) = f(d_1, \ldots, d_k, b d_{k+1}, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D(\mathbf{0}, \mathbf{1})$ for any $b \in \mathbb{R}$ $(1 \leq k \leq n-1)$
(2.9) $f(d_1, \ldots, d_n b) = f(d_1, \ldots, d_n)b$ for any $(d_1, \ldots, d_n) \in D(\mathbf{0}, \mathbf{1})$ and any $b \in \mathbb{R}$

*Proof.* The proof is carried out by induction on $n$. The case that $n = 1$ was dealt with in Proposition 2.1. Suppose, for induction, that the proposition holds for $n - 1$. For any $(d_1, \ldots, d_{n-1}) \in D(\mathbf{0}, \mathbf{1})^{n-1}$ the function $d \in D(\mathbf{0}, \mathbf{1}) \mapsto f(d_1, \ldots, d_{n-1}, d)$ satisfies the condition (2.3), so that there exists a unique function $g: D(\mathbf{0}, \mathbf{1})^{n-1} \to \mathbb{R}$ such that

(2.10) $f(d_1, \ldots, d_n) = g(d_1, \ldots, d_{n-1})d_n$ for any $(d_1, \ldots, d_n) \in D(\mathbf{0}, \mathbf{1})^n$

It is easy to see that the function $g$ satisfies conditions (2.8) and (2.9) in the case of $n$ replaced by $n - 1$. Therefore, by the induction hypothesis, there exists $a \in \mathbb{R}$ such that

(2.11) $g(d_1, \ldots, d_{n-1}) = ad_1 \ldots d_{n-1}$ for any $(d_1, \ldots, d_{n-1}) \in D(\mathbf{0}, \mathbf{1})^{n-1}$

The desired conclusion now follows from (2.10) and (2.11). ■

The reader should note again that $a$ in the above proposition is determined uniquely by the general super-Kock axiom.

Just as the general Kock axiom paved the way for the introduction of microlinear spaces, its super version invokes the notion of a *supermicrolinear space,* which is by definition a space $M$ satisfying the following condition:

(2.12) For any good finite limit diagram of Weil superalgebras with its limit $W$, the diagram obtained by taking $\mathrm{Spec}_{\mathbb{R}}$ and then exponentiating over $M$ is a limit diagram with its limit $M^{\mathrm{Spec}_{\mathbb{R}} W}$.

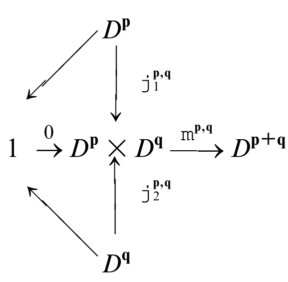
The following proposition guarantees that we have many supermicrolinear spaces.

*Proposition 2.3.* (1) $\mathbb{R}_{\mathbf{e}}$ and $\mathbb{R}_{\mathbf{o}}$ are supermicrolinear spaces.

(2) The class of supermicrolinear spaces is closed under limits and exponentiation by an arbitrary space.

*Proof.* Statement (1) follows directly from axiom (2.1), while Statement (2) can be established in the way as in Lavendhomme (1996, §§2.3, Proposition 1). ■

*Proposition 2.4.* The diagram




is a quasi-colimit diagram of supersmall objects, where

(2.13) $\mathtt{j}_1^{\mathbf{p},\mathbf{q}}(d) = (d, 0)$ for any $d \in D^{\mathbf{p}}$
(2.14) $\mathtt{j}_2^{\mathbf{p},\mathbf{q}}(d) = (0, d)$ for any $d \in D^{\mathbf{q}}$
(2.15) $\mathtt{m}^{\mathbf{p},\mathbf{q}}(d_1, d_2) = d_1 d_2$ for any $(d_1, d_2) \in D^{\mathbf{p}} \times D^{\mathbf{q}}$

*Proof.* As in Lavendhomme (1996, §2.2, Proposition 7). ■

*Corollary 2.5.* Let $M$ be a supermicrolinear space and $m \in M$. Let $\gamma$ be a function from $D^{\mathbf{p}} \times D^{\mathbf{q}}$ to $M$ such that $\gamma(d_1, 0) = \gamma(0, d_2) = m$ for any $d_1 \in D^{\mathbf{p}}$ and any $d_2 \in D^{\mathbf{q}}$. Then there exists a unique function $\delta: D^{\mathbf{p}+\mathbf{q}} \to M$ such that $\gamma(d_1, d_2) = \delta(d_1 d_2)$ for any $(d_1, d_2) \in D^{\mathbf{p}} \times D^{\mathbf{q}}$.

*Proposition 2.6.* The diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D^{\mathbf{q}} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i_2^{\mathbf{p},\mathbf{q}}} \\ D^{\mathbf{p}} & \xrightarrow[i_1^{\mathbf{p},\mathbf{q}}]{} & D(\mathbf{p}, \mathbf{q}) \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D(\mathbf{0}, \mathbf{1}) \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i_2^{(\mathbf{0},\mathbf{1})^2}} \\ D(\mathbf{0}, \mathbf{1}) & \xrightarrow[i_1^{(\mathbf{0},\mathbf{1})^2}]{} & D(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}) \end{array}$$

are quasi-colimit diagrams of supersmall objects, where

(2.16) $i_1^{\mathbf{p},\mathbf{q}}(d) = (d, 0)$ for any $d \in D^{\mathbf{p}}$
(2.17) $i_2^{\mathbf{p},\mathbf{q}}(d) = (0, d)$ for any $d \in D^{\mathbf{q}}$
(2.18) $i_1^{(\mathbf{0},\mathbf{1})^2}(d_1, d_2) = (d_1, d_2, 0, 0)$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$
(2.19) $i_2^{(\mathbf{0},\mathbf{1})^2}(d_1, d_2) = (0, 0, d_1, d_2)$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$

*Proof.* As in Lavendhomme (1996, §2.2, Proposition 6). ■

We will often denote $i_k^{\mathbf{0},\mathbf{0}}$ and $i_k^{\mathbf{1},\mathbf{1}}$ by $i_k^{\mathbf{0}^2}$ and $i_k^{\mathbf{1}^2}$, respectively ($k = 1, 2$).

*Corollary 2.7.* Let $M$ be a supermicrolinear space and $m \in M$. For any functions $\gamma_1: D^{\mathbf{p}} \to M$ and $\gamma_2: D^{\mathbf{q}} \to M$ with $\gamma_1(0) = \gamma_2(0) = m$, there exists a unique function $l_{(\gamma_1,\gamma_2)}^{\mathbf{p},\mathbf{q}}: D(\mathbf{p}, \mathbf{q}) \to M$ such that $l_{(\gamma_1,\gamma_2)}^{\mathbf{p},\mathbf{q}} \circ i_1^{\mathbf{p},\mathbf{q}} = \gamma_1$ and $l_{(\gamma_1,\gamma_2)}^{\mathbf{p},\mathbf{q}} \circ i_2^{\mathbf{p},\mathbf{q}} = \gamma_2$. For any functions $\delta_1, \delta_2: D(\mathbf{0}, \mathbf{1}) \to M$ with $\delta_1(0, 0) = \delta_2(0, 0) = m$, there exists a unique function $l_{(\delta_1,\delta_2)}^{(\mathbf{0},\mathbf{1})^2}: D(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}) \to M$ such that $l_{(\delta_1,\delta_2)}^{(\mathbf{0},\mathbf{1})^2} \circ i_1^{(\mathbf{0},\mathbf{1})^2} = \delta_1$ and $l_{(\delta_1,\delta_2)}^{(\mathbf{0},\mathbf{1})^2} \circ i_2^{(\mathbf{0},\mathbf{1})^2} = \delta_2$.

We will often denote $l_{(\gamma_1,\gamma_2)}^{\mathbf{0},\mathbf{0}}$ and $l_{(\gamma_1,\gamma_2)}^{\mathbf{1},\mathbf{1}}$ by $l_{(\gamma_1,\gamma_2)}^{\mathbf{0}^2}$ and $l_{(\gamma_1,\gamma_2)}^{\mathbf{1}^2}$, respectively.

## 3. DIFFERENTIAL CALCULUS

The super version of the Kock–Lawvere axiom, which is subsumed under the super version of the general Kock axiom discussed in the previous section, goes as follows:

(3.1) For any function $f: D \to \mathbb{R}$, there exists a unique $b \in \mathbb{R}$ such that $f(d) = f(0) + bd$ for any $d \in D$.

(3.2) For any function $g: \underline{D} \to \mathbb{R}$, there exists a unique $c \in \mathbb{R}$ such that $g(d) = g(0) + cd$ for any $d \in \underline{D}$.

Axioms (3.1) and (3.2) are equivalent to the following two axioms:

(3.3) For any function $f: D \to \mathbb{R}$, there exists a unique $b' \in \mathbb{R}$ such that $f(d) = f(0) + db'$ for any $d \in D$.

(3.4) For any function $g: \underline{D} \to \mathbb{R}$, there exists a unique $c' \in \mathbb{R}$ such that $g(d) = g(0) + dc'$ for any $d \in \underline{D}$.

These axioms as a whole are called the *super-Kock–Lawvere axiom*. The main objective of this section is to discuss some consequences of this axiom without assuming the general super-Kock axiom. It is easy to see the following:

*Proposition 3.1.* The relation between $b$ in (3.1) and $b'$ in (3.3) is simple enough; $b = b'$ for the same function $f: D \to \mathbb{R}$. The relation between $c$ in (3.2) and $c'$ in (3.4) is a bit less simple; $c'_{\mathbf{e}} = c_{\mathbf{e}}$ and $c'_{\mathbf{o}} = -c_{\mathbf{o}}$ for the same function $g: \underline{D} \to \mathbb{R}$.

Given a function $f: \mathbb{R}_{\mathbf{e}} \to \mathbb{R}$ and $a \in \mathbb{R}_{\mathbf{e}}$, by one of the equivalent axioms (3.1) and (3.3), there exist unique $(\overrightarrow{\mathbf{D}}_{\mathbf{e}} f)(a) \in \mathbb{R}$ and unique $(f\overleftarrow{\mathbf{D}}_{\mathbf{e}})(a) \in \mathbb{R}$ such that for any $d \in D$,

(3.5) $f(a + d) = f(a) + d(\overrightarrow{\mathbf{D}}_{\mathbf{e}} f)(a)$

(3.6) $f(a + d) = f(a) + (f\overleftarrow{\mathbf{D}}_{\mathbf{e}})(a)d$

The functions $a \in \mathbb{R}_{\mathbf{e}} \mapsto (\overrightarrow{\mathbf{D}}_{\mathbf{e}} f)(a)$ and $a \in \mathbb{R}_{\mathbf{e}} \mapsto (\overleftarrow{\mathbf{D}}_{\mathbf{e}} f)(a)$ are denoted by $\overrightarrow{\mathbf{D}}_{\mathbf{e}} f$ and $f\overleftarrow{\mathbf{D}}_{\mathbf{e}}$, respectively. Since they coincide by Proposition 3.1, they are often denoted unambiguously by $\mathbf{D}_{\mathbf{e}} f$.

*Proposition 3.2.* Let $f$ and $g$ be functions from $\mathbb{R}_{\mathbf{e}}$ to $\mathbb{R}$. Let $a \in \mathbb{R}$. Then we have

(3.7) $\mathbf{D}_{\mathbf{e}}(f + g) = \mathbf{D}_{\mathbf{e}} f + \mathbf{D}_{\mathbf{e}} g$

(3.8) $\mathbf{D}_{\mathbf{e}}(af) = a(\mathbf{D}_{\mathbf{e}} f)$

(3.9) $\mathbf{D}_{\mathbf{e}}(fa) = (\mathbf{D}_{\mathbf{e}} f)a$

(3.10) $\mathbf{D}_{\mathbf{e}}(fg) = (\mathbf{D}_{\mathbf{e}} f)g + f(\mathbf{D}_{\mathbf{e}} g)$

In short, $\mathbf{D}_{\mathbf{e}}$ is an even (both left- and right-) $\mathbb{R}$-derivation on the $\mathbb{R}$-superalgebra of functions from $\mathbb{R}_{\mathbf{e}}$ to $\mathbb{R}$.

*Proof.* As in Lavendhomme (1996, §1.2, Proposition 1). ■

Given a function $f: \mathbb{R}_{\mathbf{o}} \to \mathbb{R}$ and $a \in \mathbb{R}_{\mathbf{o}}$, by one of the equivalent axioms (3.2) and (3.4), there exist unique $(\overrightarrow{\mathbf{D}}_{\mathbf{o}} f)(a) \in \mathbb{R}$ and unique $(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})(a) \in \mathbb{R}$ such that for any $d \in \underline{D}$,

$$(3.11)\quad f(a+d)=f(a)+d(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)(a)$$
$$(3.12)\quad f(a+d)=f(a)+(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})(a)d$$

The functions $a\in\mathbb{R}_{\mathrm{o}}\mapsto(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)(a)$ and $a\in\mathbb{R}_{\mathbf{o}}\mapsto(\overleftarrow{\mathbf{D}}_{\mathbf{o}}f)(a)$ are denoted by $\overrightarrow{\mathbf{D}}_{\mathbf{o}}f$ and $f\overleftarrow{\mathbf{D}}_{\mathbf{o}}$, respectively.

*Proposition 3.2.* Let $f$ and $g$ be functions from $\mathbb{R}_{\mathbf{o}}$ to $\mathbb{R}$. Let $a\in\mathbb{R}$. Then we have

$$(3.13)\quad \overrightarrow{\mathbf{D}}_{\mathbf{o}}(f+g)=\overrightarrow{\mathbf{D}}_{\mathbf{o}}f+\overrightarrow{\mathbf{D}}_{\mathbf{o}}g$$
$$(3.14)\quad (f+g)\overleftarrow{\mathbf{D}}_{\mathbf{o}}=f\overleftarrow{\mathbf{D}}_{\mathbf{o}}+g\overleftarrow{\mathbf{D}}_{\mathbf{o}}$$
$$(3.15)\quad (af)\overleftarrow{\mathbf{D}}_{\mathbf{o}}=a(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})$$
$$(3.16)\quad \overrightarrow{\mathbf{D}}_{\mathbf{o}}(fa)=(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)a$$
$$(3.17)\quad \overrightarrow{\mathbf{D}}_{\mathbf{o}}(fg)=(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)g+(-1)^{|f|}f(\overrightarrow{\mathbf{D}}_{\mathbf{o}}g)\quad\text{provided that } f \text{ is pure}$$
$$(3.18)\quad (fg)\overleftarrow{\mathbf{D}}_{\mathbf{o}}=(-1)^{|g|}(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})g+f(g\overleftarrow{\mathbf{D}}_{\mathbf{o}})\quad\text{provided that } g \text{ is pure}$$

In short, $\overrightarrow{\mathbf{D}}_{\mathbf{o}}$ is an odd $\mathbb{R}$-left-derivation on the $\mathbb{R}$-superalgebra of functions from $\mathbb{R}_{\mathbf{o}}$ to $\mathbb{R}$, while $\overleftarrow{\mathbf{D}}_{\mathbf{o}}$ is an odd $\mathbb{R}$-right-derivation on it.

*Proof.* As in Lavendhomme (1996, §1.2, Proposition 1). ∎

*Proposition 3.4.* Let $f$ be a function from $\mathbb{R}_{\mathrm{o}}$ to $\mathbb{R}$. Then

$$(3.19)\quad \overrightarrow{\mathbf{D}}_{\mathbf{o}}(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)=\overrightarrow{\mathbf{D}}_{\mathbf{o}}(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})=(\overrightarrow{\mathbf{D}}_{\mathbf{o}}f)\overleftarrow{\mathbf{D}}_{\mathbf{o}}=(f\overleftarrow{\mathbf{D}}_{\mathbf{o}})\overleftarrow{\mathbf{D}}_{\mathbf{o}}=0$$

*Proof.* Since $\overline{D}=\mathbb{R}_{\mathbf{o}}$, this follows directly from one of the equivalent axioms (3.2) and (3.4). ∎

Now we would like to discuss a simple variant of Taylor's formula for a function $f\colon \mathbb{R}^{\mathbf{p}_1}\times\cdots\times\mathbb{R}^{\mathbf{p}_n}\to\mathbb{R}$ with $(\mathbf{p}_1,\ldots,\mathbf{p}_n)\in\mathbb{Z}_2^n$. We denote by $\overleftarrow{\partial}/\partial x_i$ the operator $\overleftarrow{\mathbf{D}}_{\mathbf{e}}$ or $\overleftarrow{\mathbf{D}}_{\mathbf{o}}$ with respect to the $i$th component according as $\mathbf{p}_i$ is 0 or 1 $(1\le i\le n)$. The formula goes as follows:

*Theorem 3.5.* Let $\underline{a}\in\mathbb{R}^{\mathbf{p}_1}\times\cdots\times\mathbb{R}^{\mathbf{p}_n}$. Then there exists unique $b_{k,i_1\ldots i_k}\in\mathbb{R}$ for each $k$ $(0\le k\le n)$ and each sequence $1\le i_1<\cdots<i_k\le n$ such that for any $\underline{d}=(d_1,\ldots,d_n)\in D^{\mathbf{p}_1}\times\cdots\times D^{\mathbf{p}_n}$,

$$(3.20)\quad f(\underline{a}+\underline{d})=a_0+\sum_{i=1}^{n}b_{1,i}d_i+\sum_{i_1<i_2}b_{2,i_1i_2}d_{i_1}d_{i_2}+\cdots+\sum_{1\le i_1<\cdots<i_k\le n}b_{k,i_1\ldots i_k}d_{i_1}\ldots d_{i_k}+\cdots+b_{n,1\ldots n}d_1\ldots d_n$$

More specifically, we have

$$(3.21) \quad b_{k,i_1 \ldots i_k} = \left( f \frac{\overleftarrow{\partial}}{\partial x_k} \cdots \frac{\overleftarrow{\partial}}{\partial x_1} \right)(\underline{a})$$

*Proof.* As in Lavendhomme (1996, §§1, 2.2). ■

## 4. GRADED TANGENCY

Let $M$ be a supermicrolinear space and $m \in M$. These entities shall be fixed throughout this and the next sections. A *supervector tangent to M at m* is a mapping $t$: $D(\mathbf{0}, \mathbf{1}) \to M$ with $t(0, 0) = m$. Now we would like to endow the set $\mathrm{T}_m M$ of tangent supervectors to $M$ at $m$ with an $\mathbb{R}$-supermodule structure. The set $\mathbf{T}_m M$ is called the *graded tangent space of M at m*. The left product $a \cdot t$ of $t \in \mathbf{T}_m M$ by $a \in \mathbb{R}$ and the right product $t \cdot b$ of $t$ by $b \in \mathbb{R}$ are defined by the following formulas:

$$(4.1) \quad (a \cdot t)(d) = t(da)$$
$$(4.2) \quad (t \cdot b)(d) = t(bd)$$

for any $d \in D(\mathbf{0}, \mathbf{1})$. Given $t_1, t_2 \in \mathbf{T}_m M$, their sum $t_1 + t_2$ is defined to be

$$(4.3) \quad (t_1 + t_2)(d) = l_{(t_1,t_2)}^{(\mathbf{0},\mathbf{1})^2}(d, d)$$

for any $d \in D(\mathbf{0}, \mathbf{1})$.

*Proposition 4.1.* With the above operations the set $\mathbf{T}_m M$ is an $\mathbb{R}$-bimodule.

*Proof.* As in Lavendhomme (1996, §3.1, Proposition 1). ■

*Proposition 4.2.* The $\mathbb{R}$-bimodule $\mathbf{T}_m M$ is Euclidean in the sense that it satisfies the following conditions:

(4.4) For any function $f$: $D \to \mathbf{T}_m M$, there exists a unique $t \in \mathbf{T}_m M$ such that $f(d) = f(\underline{0}) + d \cdot t$ for any $d \in D$.

(4.5) For any function $f$: $D \to \mathbf{T}_m M$, there exists a unique $t \in \mathbf{T}_m M$ such that $f(d) = f(0) + d \cdot t$ for any $d \in \underline{D}$.

*Proof.* As in Lavendhomme (1996, §§3.1, Proposition 3.2). ■

Now we define the *even tangent space* $\mathbf{T}_m^{\mathbf{0}} M$ *of M at m* to be the set of functions $t$: $D \to M$ with $t(0) = m$. It is endowed with a left $\mathbb{R}_{\mathbf{e}}$-module structure by decreeing that for any $a \in \mathbb{R}_{\mathbf{e}}$, any $t, t_1, t_2 \in \mathbf{T}_m^{\mathbf{0}} M$, and any $d \in D$,

$$(4.6) \quad (a \cdot t)(d) = t(ad)$$
$$(4.7) \quad (t_1 + t_2)(d) = l_{(t_1,t_2)}^{\mathbf{0}^2}(d, d)$$

*Proposition 4.3.* With the above operations the set $\mathbf{T}^{\mathbf{0}}_m M$ is a left $\mathbb{R}_{\mathbf{e}}$-module.

*Proof.* As in Lavendhomme (1996, §3.1, Proposition 1). ■

Similarly we define the *odd tangent space* $\mathbf{T}^{\mathbf{1}}_m M$ *of* $M$ *at* $m$ to be the set of functions $t$: $\overline{D} \to M$ with $t(0) = m$. It is endowed with an $\mathbb{R}_{\mathbf{e}}$-module structure by decreeing that for any $a \in \mathbb{R}_{\mathbf{e}}$, any $t$, $t_1$, $t_2 \in \mathbf{T}^{\mathbf{1}}_m M$, and any $d \in \overline{D}$,

$$(4.8) \quad (a \cdot t)(d) = t(ad)$$
$$(4.9) \quad (t_1 + t_2)(d) = l^{\mathbf{1}^2}_{(t_1,t_2)}(d, d)$$

*Proposition 4.4.* With the above operations the set $\mathbf{T}^{\mathbf{1}}_m M$ is a left $\mathbb{R}_{\mathbf{e}}$-module.

*Proof.* As in Lavendhomme (1996, §3.1, Proposition 1). ■

The injections $i^{\mathbf{0},\mathbf{1}}_1$: $D \to D(\mathbf{0}, \mathbf{1})$ and $i^{\mathbf{0},\mathbf{1}}_2$: $\overline{D} \to D(\mathbf{0}, \mathbf{1})$ induce functions $\mathfrak{p}_{\mathbf{e}}$: $\mathbf{T}_m M \to \mathbf{T}^{\mathbf{0}}_m M$ and $\mathfrak{p}_{\mathbf{o}}$: $\mathbf{T}_m M \to \mathbf{T}^{\mathbf{1}}_m M$, respectively. The projections $\mathfrak{p}^{\mathbf{0},\mathbf{1}}_1$: $D(\mathbf{0}, \mathbf{1}) \to D$ and $\mathfrak{p}^{\mathbf{0},\mathbf{1}}_2$: $D(\mathbf{0}, \mathbf{1}) \to \overline{D}$ induce functions $\mathfrak{i}_{\mathbf{e}}$: $\mathbf{T}^{\mathbf{0}}_m M \to \mathbf{T}_m M$ and $\mathfrak{i}_{\mathbf{o}}$: $\mathbf{T}^{\mathbf{1}}_m M \to \mathbf{T}_m M$, respectively. Then we have the following result:

*Lemma 4.5.* $\mathbf{T}_m M$ is a biproduct of $\mathbf{T}^{\mathbf{0}}_m M$ and $\mathbf{T}^{\mathbf{1}}_m M$ written the abelian category of left $\mathbb{R}_{\mathbf{e}}$-modules in the sense that

$$(4.10) \quad \mathfrak{p}_{\mathbf{e}} \circ \mathfrak{i}_{\mathbf{e}} = 1_{\mathbf{T}^{\mathbf{0}}_m M}$$
$$(4.11) \quad \mathfrak{p}_{\mathbf{o}} \circ \mathfrak{i}_{\mathbf{o}} = 1_{\mathbf{T}^{\mathbf{1}}_m M}$$
$$(4.12) \quad \mathfrak{i}_{\mathbf{e}} \circ \mathfrak{p}_{\mathbf{e}} + \mathfrak{i}_{\mathbf{o}} \circ \mathfrak{p}_{\mathbf{o}} = 1_{\mathbf{T}_m M}$$

*Proof.* Trivially (4.10) and (4.11) obtain. To see the validity of (4.12), let $t \in \mathrm{T}_m M$. Then $(\mathfrak{i}_{\mathbf{e}} \circ \mathfrak{p}_{\mathbf{e}})(t)(d_1, d_2) = t(d_1, 0)$ and $(\mathfrak{i}_{\mathbf{o}} \circ \mathfrak{p}_{\mathbf{o}})(t)(d_1, d_2) = t(0, d_2)$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$. Consider $\lambda$: $D(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}) \to M$ such that $\lambda(d_1, d_2, d_3, d_4) = t(d_1, d_4)$ for any $(d_1, d_2, d_3, d_4) \in D(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$. Then it is easy to see that $\lambda(d_1, d_2, 0, 0) = (\mathfrak{i}_{\mathbf{e}} \circ \mathfrak{p}_{\mathbf{e}})(t)(d_1, d_2)$ and $\lambda(0, 0, d_1, d_2) = (\mathfrak{i}_{\mathbf{o}} \circ \mathfrak{p}_{\mathbf{o}})(t)(d_1, d_2)$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$. Therefore $(\mathfrak{i}_{\mathbf{e}} \circ \mathfrak{p}_{\mathbf{e}} + \mathfrak{i}_{\mathbf{o}} \circ \mathfrak{p}_{\mathbf{o}})(d_1, d_2) = \lambda(d_1, d_2, d_1, d_2) = t(d_1, d_2)$ for any $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$. This completes the proof. ■

In the following $\mathbf{T}^{\mathbf{0}}_m M$ and $\mathbf{T}^{\mathbf{1}}_m M$ are regarded as $\mathbb{R}_{\mathbf{e}}$-submodules of $\mathbf{T}_m M$ in the above sense.

*Proposition 4.6.* With operations (4.1)–(4.3) the set $\mathbf{T}_m M$ is an $\mathbb{R}$-supermodule with $(\mathbf{T}_m M)_{\mathbf{e}} = \mathbf{T}^{\mathbf{0}}_m M$ and $(\mathbf{T}_m M)_{\mathbf{o}} = \mathbf{T}^{\mathbf{1}}_m M$.

*Proof.* This follows easily from Proposition 4.1 and Lemma 4.5. ■

If $M$ is $\mathbb{R}_{\mathbf{e}}$ and $\mathbb{R}_{\mathbf{o}}$, then the $\mathbb{R}$-supermodule $\mathbf{T}_m M$ is easily seen to be canonically isomorphic to $\mathbb{R}$, where $1 \in \mathbb{R}$ corresponds to the even tangent supervector $d \in D \mapsto m + d$ or to the odd tangent supervector $d \in \overline{D} \mapsto m + d$ according as $M$ is $\mathbb{R}_e$ or $\mathbb{R}_o$.

We set $\mathbf{T}^0 M = \cup_{m' \in M} \mathbf{T}^0_{m'} M$ and $\mathbf{T}^1 M = \cup_{m' \in M} \mathbf{T}^1_{m'} M$.

A *supervector field on $M$* is a tangent supervector to $M^M$ at $1_M$, i.e., it is an assignment $X$ of an infinitesimal transformation $X_{(d_1,d_2)}: M \to M$ to each $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$. The totality of supervector fields on $M$ is denoted by $\chi(M)$. The $\mathbb{R}$-supermodule $\chi(M)$ can be decomposed into its even and odd parts, which are denoted by $\chi^0(M)$ and $\chi^1(M)$, respectively. An even supervector field on $M$ can be identified with an assignment $X$ of an infinitesimal transformation $X_d: M \to M$ to each $d \in D$ with $X_o = 1_M$, while an odd supervector field on $M$ can be reckoned as an assignment $X$ of an infinitesimal transformation $X_d: M \to M$ to each $d \in \overline{D}$ with $X_o = 1_M$.

Given two pure supervector fields $X$, $Y$ on $M$, we now define their Lie bracket $[X, Y]$ by Corollary 2.4 as follows:

(4.13) If $X \in \chi^0(M)$ and $Y \in \chi^0(M)$, then $[X, Y]$ is the unique even supervector field on $M$ such that $[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$ for any $d_1, d_2 \in D$

(4.14) If $X \in \chi^0(M)$ and $Y \in \chi^1(M)$, then $[X, Y]$ is the unique odd supervector field on $M$ such that $[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$ for any $d_1 \in D$ and any $d_2 \in \overline{D}$

(4.15) If $X \in \chi^1(M)$ and $Y \in \chi^0(M)$, then $[X, Y]$ is the unique odd supervector field on $M$ such that $[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$ for any $d_1 \in \overline{D}$ and any $d_2 \in D$

(4.16) If $X \in \chi^1(M)$ and $Y \in \chi^1(M)$, then $[X, Y]$ is the unique even supervector field on $M$ such that $[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$ for any $d_1, d_2 \in \overline{D}$

Once the Lie bracket of any two pure supervector fields on $M$ is defined, we can define the Lie bracket $[X, Y]$ of two nonpure supervector fields $X$, $Y$ on $M$ by the following formula:

(4.17) $[X, Y] = [X_\mathbf{e}, Y_\mathbf{e}] + [X_\mathbf{e}, Y_\mathbf{o}] + [X_\mathbf{o}, Y_\mathbf{e}] + [X_\mathbf{o}, Y_\mathbf{o}]$

The proof of the following theorem is relegated to the succeeding section.

*Theorem 4.6.* $\chi(M)$ is a Lie superalgebra.

## 5. SUPERMICROSQUARES AND SUPERMICROCUBES

The main objective of this section is to discuss fundamental properties of microsquares and microcubes in our supercontext and apply them to Lie brackets of supervector fields.

A *supermicrosquare of type* $(\mathbf{p}, \mathbf{q})$ *on* $M$ *at* $m$ is a function $\alpha$ from $D^{\mathbf{p}} \times D^{\mathbf{q}}$ to $M$ with $\alpha(0, 0) = m$. The totality of supermicrosquares of type (p, q) on $M$ at $m$ is denoted by $\mathbf{T}_m^{\mathbf{p},\mathbf{q}}M$, and we set $\mathbf{T}^{\mathbf{p},\mathbf{q}}M = \cup_{m' \in M}\mathbf{T}_{m'}^{\mathbf{p},\mathbf{q}}M$.

*Lemma 5.1.* The diagram

$$\begin{array}{ccc} D(\mathbf{p}, \mathbf{q}) & \xrightarrow{\quad i \quad} & D^{\mathbf{p}} \times D^{\mathbf{q}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi^{\mathbf{p},\mathbf{q}}} \\ D^{\mathbf{p}} \times D^{\mathbf{q}} & \xrightarrow[\varphi^{\mathbf{p},\mathbf{q}}]{} & (D^{\mathbf{p}} \times D^{\mathbf{q}}) \vee D^{\mathbf{p}+\mathbf{q}} \end{array}$$

is a quasi-colimit diagram of supersmall objects, where

(5.1) $(D^{\mathbf{p}} \times D^{\mathbf{q}}) \vee D^{\mathbf{p}+\mathbf{q}} = \{(d_1, d_2, d_3) \in D^{\mathbf{p}} \times D^{\mathbf{q}} \times D^{\mathbf{p}+\mathbf{q}} \mid d_1 d_3 = d_2 d_3 = 0\}$

(5.2) $\varphi^{\mathbf{p},\mathbf{q}}(d_1, d_2) = (d_1, d_2, 0)$ for any $(d_1, d_2) \in D^{\mathbf{p}} \times D^{\mathbf{q}}$

(5.3) $\psi^{\mathbf{p},\mathbf{q}}(d_1, d_2) = (d_1, d_2, d_1 d_2)$ for any $(d_1, d_2) \in D^{\mathbf{p}} \times D^{\mathbf{q}}$

*Proof.* As in Lavendhomme (1996, §3.4, pp. 92–93, Lemma). ■

*Proposition 5.2.* For any $\alpha_1, \alpha_2 \in \mathbf{T}^{\mathbf{p},\mathbf{q}}M$, if $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, then there exists a unique function $\mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}: (D^{\mathbf{p}} \times D^{\mathbf{q}}) \vee D^{\mathbf{p}+\mathbf{q}} \to M$ such that $\mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)} \circ \varphi^{\mathbf{p},\mathbf{q}} = \alpha_1$ and $\mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)} \circ \psi^{\mathbf{p},\mathbf{q}} = \alpha_2$. In this case we define a pure tangent supervector $\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1$ to $M$ as follows, where it is even if $\mathbf{p} + \mathbf{q} = \mathbf{0}$ and odd if $\mathbf{p} + \mathbf{q} = \mathbf{1}$:

$$\left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1\right)(d) = \mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(0, 0, d) \quad \text{for any} \quad d \in D^{\mathbf{p}+\mathbf{q}} \tag{5.4}$$

*Proof.* This follows from Lemma 5.1. ■

*Proposition 5.3.* For any $\alpha_1, \alpha_2 \in \mathbf{T}_m^{\mathbf{p},\mathbf{q}}M$ with $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, we have

$$\alpha_1 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_2 = -\left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1\right) \tag{5.5}$$

*Proof.* We define $\mathtt{h}: (D^{\mathbf{p}} \times D^{\mathbf{q}}) \vee D^{\mathbf{p}+\mathbf{q}} \to M$ as follows:

(5.6) $\mathtt{h}(d_1, d_2, d_3) = \mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(d_1, d_2, d_1 d_2 - d_3)$ for any $(d_1, d_2, d_3) \in (D^{\mathbf{p}} \times D^{\mathbf{q}}) \vee D^{\mathbf{p}+\mathbf{q}}$

Then it is easy to see that $\mathtt{h} \circ \varphi^{\mathbf{p},\mathbf{q}} = \alpha_2$ and $\mathtt{h} \circ \psi^{\mathbf{p},\mathbf{q}} = \alpha_1$. Therefore $\mathtt{h} = \mathrm{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_2,\alpha_1)}$, which implies (5.5). ■

For any $\alpha \in \mathbf{T}_m^{\mathbf{p},\mathbf{q}}M$, we define $\Sigma(\alpha) \in \mathbf{T}_m^{\mathbf{p},\mathbf{q}}M$ to be

$$(5.7) \quad \Sigma(\alpha)(d_1, d_2) = \alpha(d_2, d_1) \quad \text{for any} \quad (d_1, d_2) \in D^{\mathbf{q}} \times D^{\mathbf{p}}$$

The following proposition, which reveals the underlying structure of the graded anticommutativity of supervector fields with respect to Lie brackets, strongly reminds us that we now dwell in a world where commutativity is no longer a dictum.

*Proposition 5.4.* For any $\alpha_1, \alpha_2 \in \mathbf{T}_m^{\mathbf{p},\mathbf{q}}M$ with $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, we have

$$(5.8) \quad \Sigma(\alpha_1)|_{D(\mathbf{q},\mathbf{p})} = \Sigma(\alpha_2)|_{D(\mathbf{q},\mathbf{p})}$$

$$(5.9) \quad \Sigma(\alpha_2) \underset{\mathbf{q},\,\mathbf{p}}{\dot{-}} \Sigma(\alpha_1) = (-1)^{\mathbf{pq}}\left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_2\right)$$

*Proof.* Let us define $\mathtt{h}: (D^{\mathbf{q}} \times D^{\mathbf{p}}) \vee D^{\mathbf{p}+\mathbf{q}} \to M$ as follows:

$$(5.10) \quad \mathtt{h}(d_1, d_2, d_3) = \mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(d_2, d_1, (-1)^{\mathbf{pq}}d_3) \text{ for any } (d_1, d_2, d_3) \in (D^{\mathbf{q}} \times D^{\mathbf{p}}) \vee D^{\mathbf{p}+\mathbf{q}}$$

Then it is easy to see that $\mathtt{h} \circ \varphi^{\mathbf{q},\mathbf{p}} = \Sigma(\alpha_1)$ and $\mathtt{h} \circ \psi^{\mathbf{q},\mathbf{p}} = \Sigma(\alpha_2)$, whence (5.9) follows. ■

Now we discuss a super version of the microcube. A *supermicrocube of type* $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ *on* $M$ *at* $m$ is a function $\gamma$ from $D^{\mathbf{p}} \times D^{\mathbf{q}} \times D^{\mathbf{r}}$ to $M$ with $\alpha(0, 0, 0) = m$. The totality of supermicrocubes of type $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ on $M$ at $m$ is denoted by $\mathbf{T}_m^{\mathbf{p},\mathbf{q},\mathbf{r}}M$, and we set $\mathbf{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}M = \cup_{m' \in M}\mathbf{T}_{m'}^{\mathbf{p},\mathbf{q},\mathbf{r}}M$.

Now we relativize the partial binary operation $\underset{\mathbf{q},\,\mathbf{r}}{\dot{-}}$ to $\mathbf{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}M$. As we discussed in Nishimura (1998c, §§1.3), we can do so by regarding $\mathbf{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}M$ either as $\mathbf{T}^{\mathbf{p}}(\mathbf{T}^{\mathbf{q},\mathbf{r}}M)$ or as $\mathbf{T}^{\mathbf{q},\mathbf{r}}(\mathbf{T}^{\mathbf{p}}M)$. Fortunately both approaches result in the same partial operation $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{1}{-}}$; given $\gamma_1, \gamma_2 \in \mathbf{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}M$, $\gamma_2 \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{1}{-}} \gamma_1$ is defined iff $\gamma_1|_{D^{\mathbf{p}} \times D(\mathbf{q},\mathbf{r})} = \gamma_2|_{D^{\mathbf{p}} \times D(\mathbf{q},\mathbf{r})}$, in which it is a supermicrosquare of type $(\mathbf{p}, \mathbf{q} + \mathbf{r})$ on $M$.

Let $\mathfrak{Perm}_3$ denote the group of permutations of the set $\{1, 2, 3\}$. Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}M$ and $\rho \in \mathfrak{Perm}_3$, we define $\Sigma_\rho(\gamma) \in \mathbf{T}^{\mathbf{p}_{\rho^{-1}(1)},\mathbf{p}_{\rho^{-1}(2)},\mathbf{p}_{\rho^{-1}(3)}}M$ as follows:

$$(5.11) \quad \Sigma_\rho(\gamma)(d_1, d_2, d_3) = \gamma(d_{\rho(1)}, d_{\rho(2)}, d_{\rho(3)}) \text{ for any } (d_1, d_2, d_3) \in D^{\mathbf{p}_{\rho^{-1}(1)}} \times D^{\mathbf{p}_{\rho^{-1}(2)}} \times D^{\mathbf{p}_{\rho^{-1}(3)}}$$

Now we define partial binary operations $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{2}{-}}$ and $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{3}{-}}$ in $\mathbf{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}M$ as follows:

(5.12) $\gamma_2 \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{2} \gamma_1$ is defined iff

$\Sigma_{(132)}(\gamma_2) \xrightarrow[\mathbf{q},\,\mathbf{r},\,\mathbf{p}]{\mathrm{i}} \Sigma_{(132)}(\gamma_1)$ is defined, in which the former is defined to be the latter.

(5.13) $\gamma_2 \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{3} \gamma_1$ is defined iff

$\Sigma_{(132)}(\gamma_2) \xrightarrow[\mathbf{r},\,\mathbf{p},\,\mathbf{q}]{\mathrm{i}} \Sigma_{(123)}(\gamma_1)$ is defined, in which the former is defined to be the latter.

The following theorem reveals the underlying structure of the graded Jacobi identity of Lie brackets of supervector fields.

*Theorem 5.5.* Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in \mathbf{T}_m^{\mathbf{p},\mathbf{q},\mathbf{r}}M$. Let us suppose that the following three expressions are well defined:

$$\left(\gamma_{123} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{\mathrm{i}} \gamma_{132}\right) \xrightarrow[\mathbf{p},\,\mathbf{q}+\mathbf{r}]{\cdot} \left(\gamma_{231} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{\mathrm{i}} \gamma_{321}\right) \tag{5.14}$$

$$\left(\gamma_{231} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{2} \gamma_{213}\right) \xrightarrow[\mathbf{q},\,\mathbf{p}+\mathbf{r}]{\cdot} \left(\gamma_{312} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{2} \gamma_{132}\right) \tag{5.15}$$

$$\left(\gamma_{312} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{3} \gamma_{321}\right) \xrightarrow[\mathbf{r},\,\mathbf{p}+\mathbf{q}]{\cdot} \left(\gamma_{123} \xrightarrow[\mathbf{p},\,\mathbf{q},\,\mathbf{r}]{3} \gamma_{213}\right) \tag{5.16}$$

Letting $\delta_1$, $\delta_2$, and $\delta_3$ denote the above three expressions in order, we have

$$\delta_1 + (-1)^{\mathbf{p}(\mathbf{q}+\mathbf{r})}\delta_2 + (-1)^{(\mathbf{p}+\mathbf{q})\mathbf{r}}\delta_3 = 0 \tag{5.17}$$

*Proof.* As in Nishimura (1997b, §3). ■

Now we apply the above theory of supermicrosquares and supermicrocubes to Lie brackets of vector fields. We denote by $\chi^{\mathbf{p},\mathbf{q}}(M)$ the totality of supermicrosquares on $M^M$ at $1_M$. We denote by $\chi^{\mathbf{p},\mathbf{q},\mathbf{r}}(M)$ the totality of supermicrocubes on $M^M$ at $1_M$. Given $X \in \chi^{\mathbf{p}}(M)$, $Y \in \chi^{\mathbf{q}}(M)$, and $Z \in \chi^{\mathbf{r}}(M)$, we define $Y * X \in \chi^{\mathbf{p},\mathbf{q}}$ and $Z * Y * X \in \chi^{\mathbf{p},\mathbf{q},\mathbf{r}}(M)$ as follows:

$$(Y * X)(d_1, d_2) = Y_{d_2} \circ X_{d_1} \quad \text{for any} \quad (d_1, d_2) \in D^{\mathbf{p}} \times D^{\mathbf{q}} \tag{5.18}$$

$$(Z * Y * X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1} \quad \text{for any} \quad (d_1, d_2, d_3) \in D^{\mathbf{p}} \times D^{\mathbf{q}} \times D^{\mathbf{r}} \tag{5.19}$$

*Proposition 5.6.* Let $X \in \chi^{\mathbf{p}}(M)$ and $Y \in \chi^{\mathbf{q}}(M)$. Then we have

$$[X, Y] = Y * X \xrightarrow[\mathbf{p},\,\mathbf{q}]{\cdot} \Sigma(X * Y) \tag{5.20}$$

*Proof.* As in Lavendhomme (1996, §3.4, Proposition 8). ■

*Theorem 5.7.* Let $X \in \chi^{\mathbf{p}}(M)$ and $Y \in \chi^{\mathbf{q}}(M)$. Then we have

(5.21) $[X, Y] = -(-1)^{\mathbf{pq}}[Y, X]$

*Proof.* We have

$$
\begin{aligned}
&[X, Y] \\
&\quad = Y * X \frac{\cdot}{\mathbf{p},\, \mathbf{q}} \Sigma(X * Y) \\
&\quad = -\left(\Sigma(X * Y) \frac{\cdot}{\mathbf{p},\, \mathbf{q}} Y * X\right) \quad \text{[Proposition 5.3]} \\
&\quad = -(-1)^{\mathbf{pq}}\left(X * Y \frac{\cdot}{\mathbf{q},\, \mathbf{p}} \Sigma(Y * X)\right) \quad \text{[Propositiona 5.4]} \\
&\quad = -(-1)^{\mathbf{pq}}[Y, X] \quad ■
\end{aligned}
$$

*Proposition 5.8.* Let $X \in \chi^{\mathbf{p}}(M)$, $Y \in \chi^{\mathbf{q}}(M)$, and $Z \in \chi^{\mathbf{r}}(M)$. Let it be the case that

(5.22) $\gamma_{123} = Z * Y * X$
(5.23) $\gamma_{132} = \Sigma_{(23)}(Y * Z * X)$
(5.24) $\gamma_{213} = \Sigma_{(12)}(Z * X * Y)$
(5.25) $\gamma_{231} = \Sigma_{(123)}(X * Z * Y)$
(5.26) $\gamma_{312} = \Sigma_{(132)}(Y * X * Z)$
(5.27) $\gamma_{321} = \Sigma_{(13)}(X * Y * Z)$

Then the right-hand sides of the following three identities are meaningful, and all the three identities hold:

(5.28) $[X, [Y, Z]]$
$$= \left(\gamma_{123} \frac{i}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{132}\right) \frac{\cdot}{\mathbf{p},\, \mathbf{q} + \mathbf{r}} \left(\gamma_{231} \frac{1}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{321}\right)$$
(5.29) $[Y, [Z, X]]$
$$= \left(\gamma_{231} \frac{2}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{213}\right) \frac{\cdot}{\mathbf{q},\, \mathbf{p} + \mathbf{r}} \left(\gamma_{312} \frac{2}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{132}\right)$$
(5.30) $[Z, [X, Y]]$
$$= \left(\gamma_{312} \frac{3}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{321}\right) \frac{\cdot}{\mathbf{r},\, \mathbf{p} + \mathbf{q}} \left(\gamma_{123} \frac{3}{\mathbf{p},\, \mathbf{q},\, \mathbf{r}} \gamma_{213}\right)$$

*Proof.* As in Nishimura (1997b, Proposition 2.7). ■

*Theorem 5.9.* Let $X \in \chi^{\mathbf{p}}(M)$, $Y \in \chi^{\mathbf{q}}(M)$, and $Z \in \chi^{\mathbf{r}}(M)$. Then

$$[X, [Y, Z]] + (-1)^{\mathbf{p}(\mathbf{q}+\mathbf{r})}[Y, [Z, X]] + (-1)^{(\mathbf{p}+\mathbf{q})\mathbf{r}}[Z, [X, Y]] = 0 \tag{5.31}$$

*Proof.* Follows from Theorem 5.5 and Proposition 5.8. ■

We conclude this section by remarking that Theorems 5.7 and 5.9 constitute a proof of Theorem 4.6.

**Note added in proof**

After finishing this paper, we got acquainted with the following paper, which should be put down as a precursor of ours. Yetter, D. N. (1988). Models for synthetic supergeometry, *Cahiers Topologie Géométrie Différentielle Catégoriques,* **29**, 87–108.

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